2020 A

Fall 2020/21

Lecture 5

- Double $\iint$ in Polar Coordinates.

Let $\Phi : (r,\theta) \mapsto (x,y)$ given by

$$x = r\cos\theta, \quad y = r\sin\theta$$

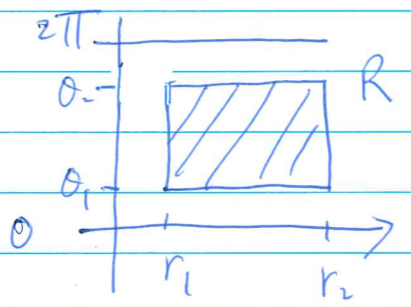
We consider Φ as a map from

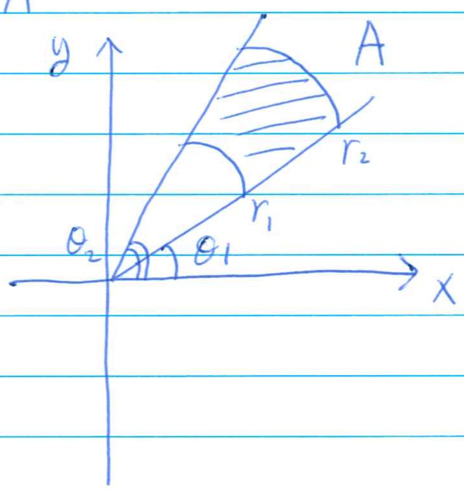
$$\{(r,\theta) : r \geq 0,\ \theta \in [0, 2\pi]\} \text{ onto } \mathbb{R}^2.$$

It is 1-1 in its interior. A rectangle

$$R = \{(r,\theta) : r_1 \leq r \leq r_2,\ \theta_1 \leq \theta \leq \theta_2\} \text{ is}$$

mapped to a "curved rectangle" A





<u>Theorem 10</u> R and A as above. Let f be a piecewise conti in A. Then

$$\iint_A f = \iint_R \hat{f}\, r .$$

Recall that whenever f is a function in (x,y), its pull-back in (r,θ) is

$$\hat{f}(r,\theta) = f(r\cos\theta, r\sin\theta).$$

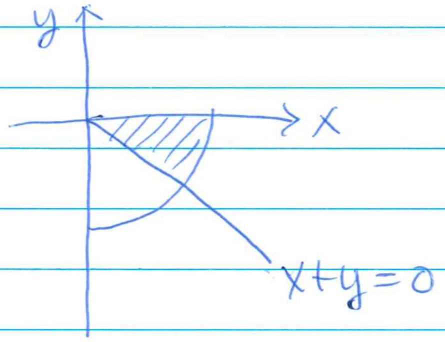
For instance, $f(x,y) = xy^2 + e^{x^2+y^2}$,

$$\hat{f}(r,\theta) = r\cos\theta\,(r\sin\theta)^2 + e^{r^2}$$

$$= r^3\cos\theta\sin^2\theta + e^{r^2}.$$

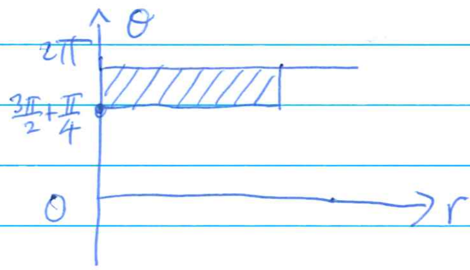
e.g. 7 (old)

$$\iint_D x\,dx\,dy$$ where D is bdd by x-axis, $x+y=0$ and the unit circle.



Using thm 10,

$$\iint_D x\,dx\,dy = \int_{\frac{3\pi}{2}+\frac{\pi}{4}}^{2\pi} \int_0 r\cos\theta\, r\,dr\,d\theta$$



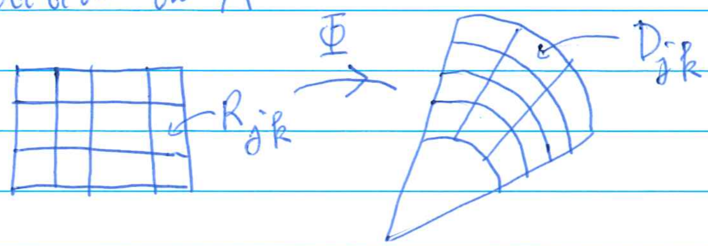
$$= \frac{1}{3}\int_{\frac{3\pi}{2}+\frac{\pi}{4}}^{2\pi} \cos\theta\,d\theta$$

$$= \frac{1}{3}\sin\theta \Big|_{\frac{3\pi}{2}+\frac{\pi}{4}}^{2\pi}$$

$$= \frac{1}{3}\frac{\sqrt{2}}{2}. \quad \#$$

Φ maps $[0,1]\times[\frac{3\pi}{2}+\frac{\pi}{4}, 2\pi]$ to D.

"Pf of Thm 10" Let P be a partition on R. Under Φ it induces "generalized partition" in A



As $\|P\| \to 0$, we believe the "norm" of the generalized partition G also $\to 0$

and the "Riemann sum"

$$S(f,\dot{Q}) \equiv \sum f(P_{jk})\,|D_{jk}| \to \iint_D f \,.$$ (Explained below)

Using the area for a sector =

$\frac{1}{2}\theta r^2$ (θ open angle)

$$|D_{jk}| = \frac{1}{2}(\theta_k - \theta_{k-1})(r_j^2 - r_{j-1}^2)$$

$$= \frac{1}{2}(r_j + r_{j-1})\,\Delta r_j\,\Delta\theta_k \,,$$

So

$$S(f,\dot{Q}) = \sum f(P_{jk})\frac{1}{2}(r_j + r_{j-1})\,\Delta r_j\,\Delta\theta_k \,.$$

As $P_{jk} \in D_{jk}$ is a tag pt which is arbitrary, we choose it to be the form $(\bar{r}_j \cos\theta_k, \bar{r}_j \sin\theta_k)$ when $\bar{r}_j = \frac{1}{2}(r_j + r_{j-1})$.

Then

$$S(f,\dot{Q}) = \sum \hat{f}(\bar{r}_j, \theta_k)\,\bar{r}_j\,\Delta r_j\,\Delta\theta_k \,.$$

If we let $g(r,\theta) = \hat{f}(r,\theta)\,r$.

$$S(f,\dot{Q}) = S(g,\dot{P})$$ when $(\bar{r}_j, \theta_k) \in R_{jk}$ is the tag. So as $\|P\| \to 0$, we have

$$S(f,\dot{Q}) \to \iint_D f$$

$$\|$$

$$S(g,\dot{P}) \to \iint_R g \,,$$ done. #

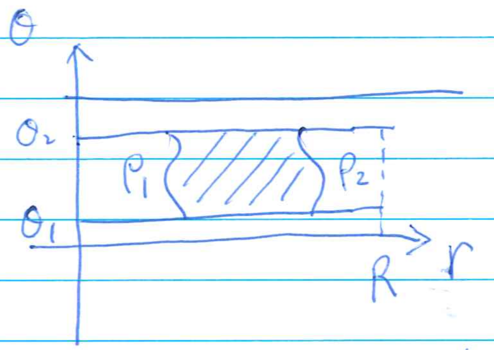
Theorem 11 Let

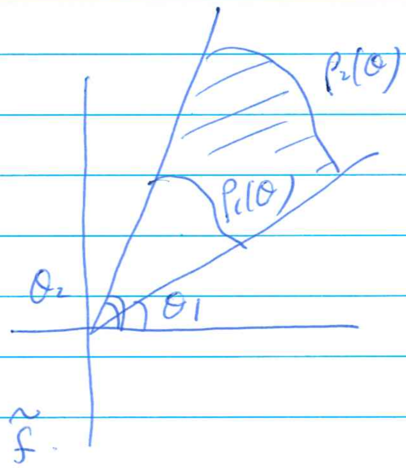
$$D = \{(x,y) : x = r\cos\theta,\ y = r\sin\theta,\ \theta\in[\theta_1,\theta_2],\ \rho_1(\theta)\le r\le\rho_2(\theta)\}$$

Let f be piecewise continuous in D. Then

$$\iint_D f = \int_{\theta_1}^{\theta_2}\int_{\rho_1(\theta)}^{\rho_2(\theta)} \hat{f}(r,\theta)\, r\,dr\,d\theta .$$

Pf:





Extend f to 0 outside D & denote it by $\tilde{f}$. Fix some large R_0 s.t. D is contained in the set

$$D_0 \equiv \{(x,y) : \cdots\ \theta\in[\theta_1,\theta_2],\ 0\le r\le R_0\}$$

Then

$$\iint_D f = \iint_{D_0} \tilde{f}$$

$$= \int_{\theta_1}^{\theta_2}\int_0^{R_0} \hat{\tilde{f}}(r,\theta)\, r\,dr\,d\theta \qquad (\text{Thm } 10)$$

$$= \int_{\theta_1}^{\theta_2}\left(\int_0^{\rho_1(\theta)} + \int_{\rho_1(\theta)}^{\rho_2(\theta)} + \int_{\rho_2(\theta)}^{R_0}\right) \hat{\tilde{f}}(r,\theta)\, r\,dr\,d\theta$$

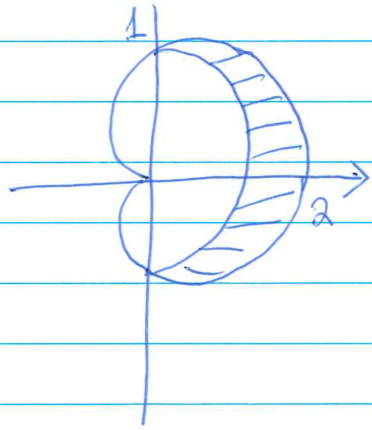
As ~~Hence~~ $\tilde{f}=0$ outside D, $\hat{f}=0$ in $\{(r,\theta): 0\le r\le \rho_1(\theta),\ \theta\in[\theta_1,\theta_2]\}$

and $\{(r,\theta): \rho_2(\theta)\le r\le R_0,\ \theta\in[\theta_1,\theta_2]\}$, we have

$$\iint_D f = \cdots$$

$$= \int_{\theta_1}^{\theta_2}\int_{\rho_1(\theta)}^{\rho_2(\theta)} \hat{f}(r,\theta)\, r\,dr\,d\theta. \ \#$$

e.g. 8 $\displaystyle\iint_D \frac{1}{\sqrt{x^2+y^2}}$ where D is the region bounded between $r=1$ and $r=1+\cos\theta$.



$$\iint_D \frac{1}{\sqrt{x^2+y^2}} = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} \frac{1}{r}\, r\,dr\,d\theta$$

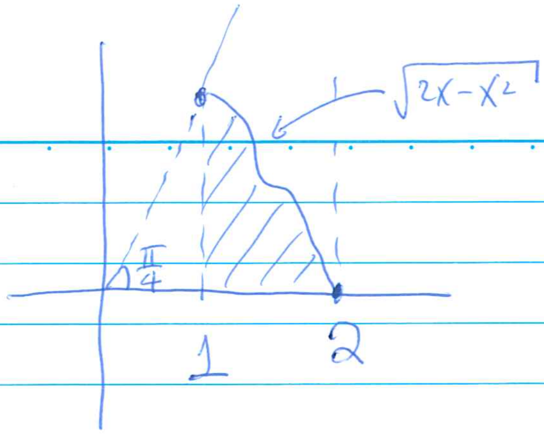
$$= 2\int_0^{\frac{\pi}{2}} (1+\cos\theta-1)\,d\theta$$

$$= 2 \ \#$$

e.g. 9. Evaluate

$$\int_1^2 \int_0^{\sqrt{2x-x^2}} y\,dy\,dx.$$

Sketch the region first



$$\int_1^2 \int_0^{\sqrt{2x-x^2}} y \, dy \, dx$$

$\rho_1(\theta) = ?$ $\quad x = 1$

$\rho_2(\theta) = ?$ $\quad y = \sqrt{2x-x^2}$

$\rho_1(\theta) = \frac{1}{\cos\theta}$

$\rho_2(\theta) = 2\cos\theta$

$$= \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\cos\theta}}^{2\cos\theta} r^2 \sin\theta \, dr \, d\theta$$

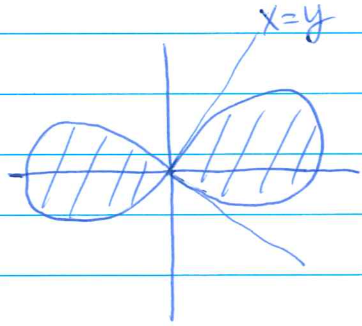
$$= \int_0^{\frac{\pi}{4}} \frac{1}{3} r^3 \Big|_{\frac{1}{\cos\theta}}^{2\cos\theta} \sin\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(8\cos^3\theta - \frac{1}{\cos^3\theta}\right) \sin\theta \, d\theta = \cdots \quad \#$$

(In fact, $y = \sqrt{2x-x^2}$ is the circle $\rho = 2\cos\theta$.)

eg. 10 Find the area enclosed by the lemniscate

$$r^2 = 4\cos 2\theta.$$



By symmetry, the area is

$$4 \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{4\cos 2\theta}} r \, dr \, d\theta$$

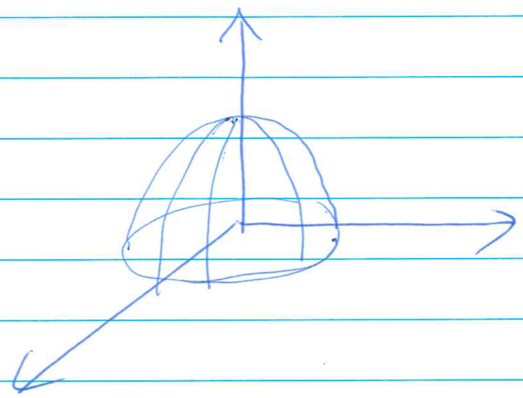
$$= 4 \int_0^{\frac{\pi}{4}} \frac{1}{2} \times 4\cos 2\theta \, d\theta$$

$$= 4 \quad \#$$

e.g. 11. Find the average height of the hemisphere of radius a.

Average height $\bar{h}$

$$= \frac{1}{|D|}\iint_D \sqrt{a^2-x^2-y^2}\, dA$$



when D is the unit disk.

$$\bar{h} = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^{a}\sqrt{a^2-r^2}\, r\,dr\,d\theta$$

$$= \frac{1}{\pi a^2}\int_0^{2\pi}\frac{1}{3}a^3\,d\theta$$

$$= \frac{2}{3}a \quad \#$$

Finally, we go back to P8?.

Call $\{D_j\}_{j=1}^{n}$ a collection of sub-regions a generalized partition of D if $D=\bigcup D_j$ and the interiors of D_j's are mutually disjoint. Denote it by P and set

$$\|P\| = \max\{\operatorname{diam} D_1, \ldots, \operatorname{diam} D_n\}.$$

(For a partition, $\|P\| = \max\{\Delta x_1, \Delta y_1, \ldots, \Delta x_n, \Delta y_n\}$ old

$$\|P\| = \max\left\{\sqrt{(\Delta x_1)^2+(\Delta y_1)^2}, \ldots, \sqrt{(\Delta x_n)^2+(\Delta y_n)^2}\right.$$

clearly, $\|P\|_{new} \to 0 \iff \|P\|_{old} \to 0$.)

Theorem 12 Let f be piecewise continu in D. Then

$$\lim_{\|P\| \to 0} S(f, \mathring{P}) = \iint_D f, \text{ where } P \text{ is}$$

a generalized partition.

Pf: For any generalized partition $P = \{D_j\}_{j=1}^n$, let

$$m_j = \min_{D_j} f, \quad M_j = \max_{D_j} f.$$

then $\quad m_j \chi_{D_j} \le f \chi_{D_j} \le M_j \chi_{D_j}$, so

$$m_j |D_j| \le \iint_{D_j} f \le M_j |D_j|$$

$$m_j \le \frac{1}{|D_j|} \iint_{D_j} f \le M_j.$$

By continuity, $\exists\, P_j \in D_j$ s.t.

$$f(P_j) = \frac{1}{|D_j|} \iint_{D_j} f.$$

Therefore,

$$\iint_D f = \sum_j \iint_{D_j} f$$

$$= \sum f(p_j) |D_j| .$$

By continuity, for $\varepsilon > 0$, $\exists\, \delta$ s.t. whenever $|x-y| < \delta$,

$$|f(x) - f(y)| < \varepsilon .$$

Hence, $\forall q_j \in D_j$, $|f(q_j) - f(p_j)| < \varepsilon$, so

$$\left| \iint_D f - S(f, \dot{P}) \right| = \left| \iint_D f - \sum f(q_j) |D_j| \right|$$

$$= \left| \sum f(p_j) |D_j| - \sum f(q_j) |D_j| \right|$$

$$= \left| \sum \left(f(p_j) - f(q_j) \right) |D_j| \right|$$

$$\leq \sum |f(p_j) - f(q_j)| \, |D_j|$$

$$< \varepsilon \sum_j |D_j|$$

$$= \varepsilon |D| \text{ , arb. small. } \#$$